New Algorithms for Learning Incoherent and Overcomplete Dictionaries

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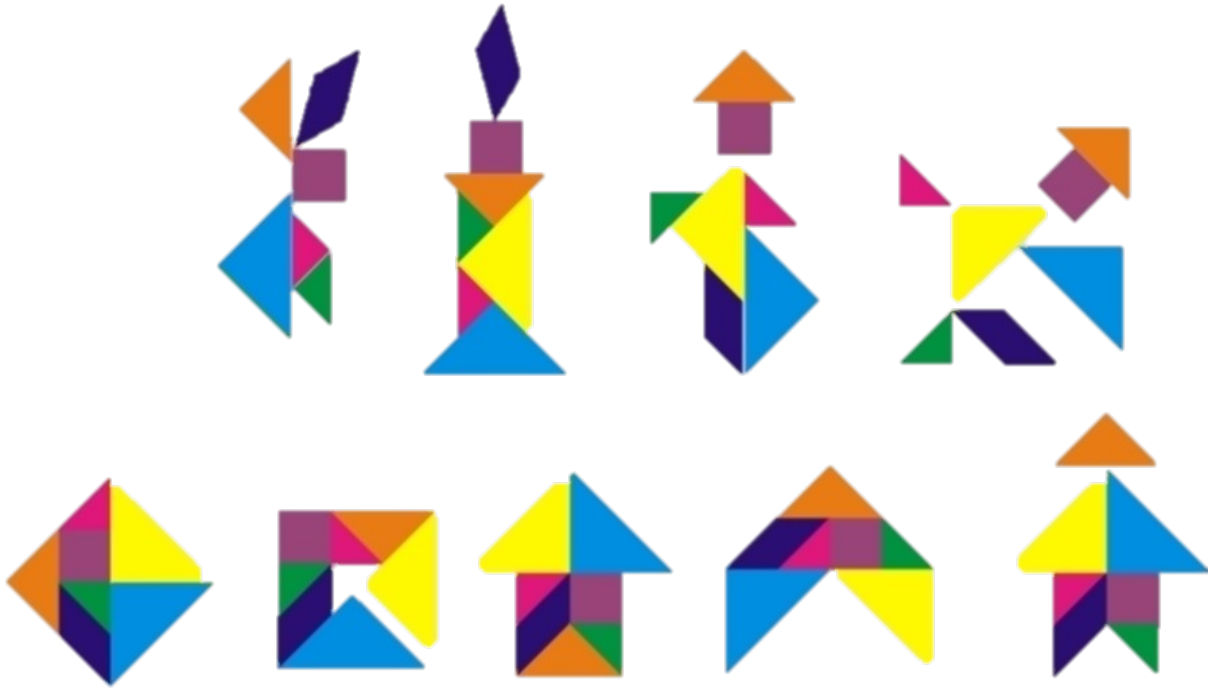
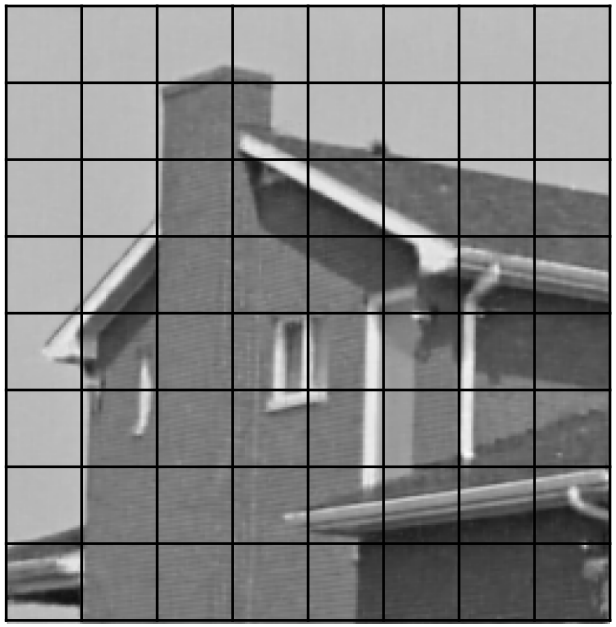
Dictionary Learning

- Simple "dictionary elements" build complicated objects



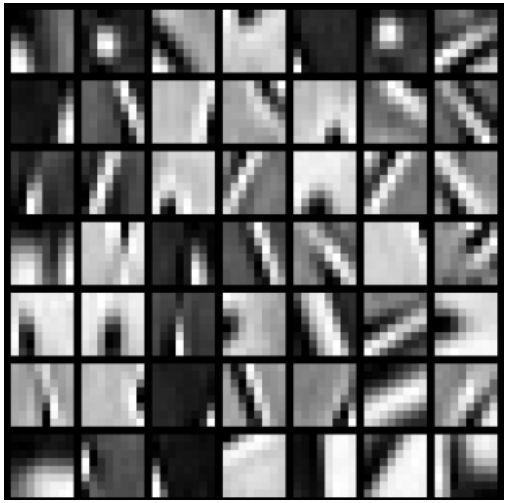
- Given the objects, can we learn the dictionary?

Why dictionary learning? [Olshausen Field '96]

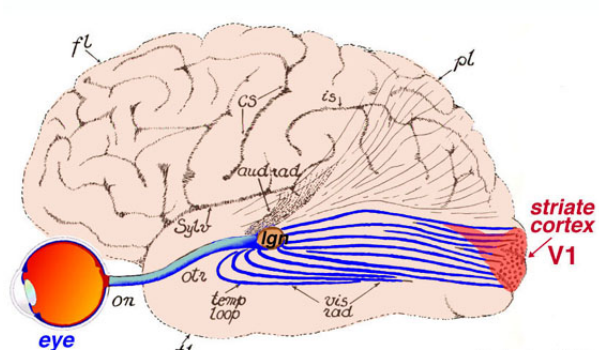


natural image patches

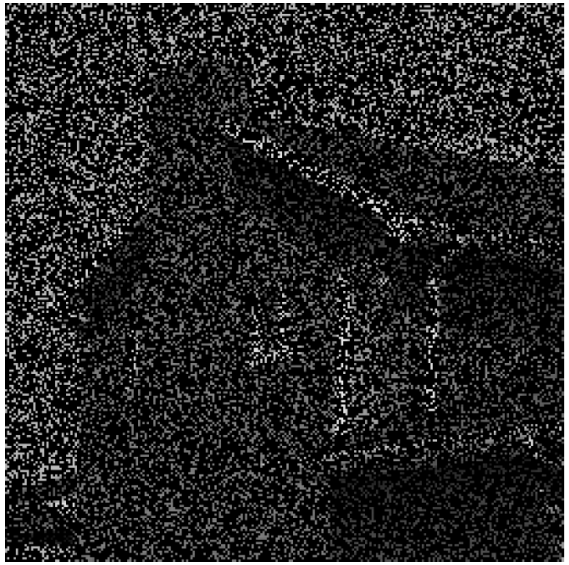
dictionary learning

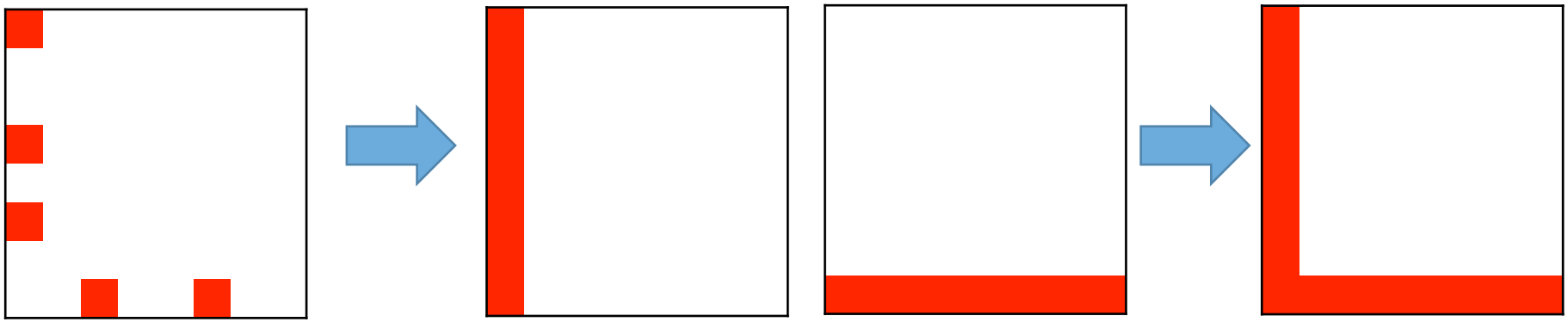


Gabor-like Filters



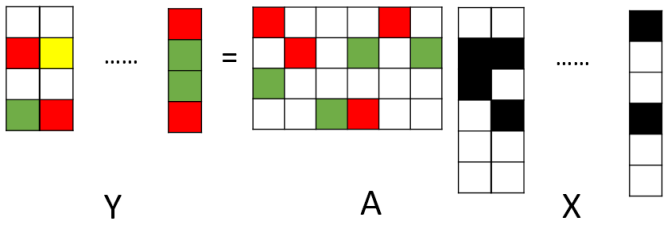
Example: Image Completion [Mairal, Elad & Sapiro '08]



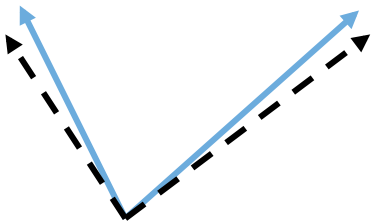


Outline

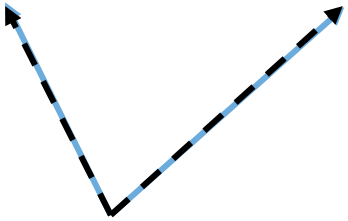
- Dictionary Learning problem



- Getting a crude estimate

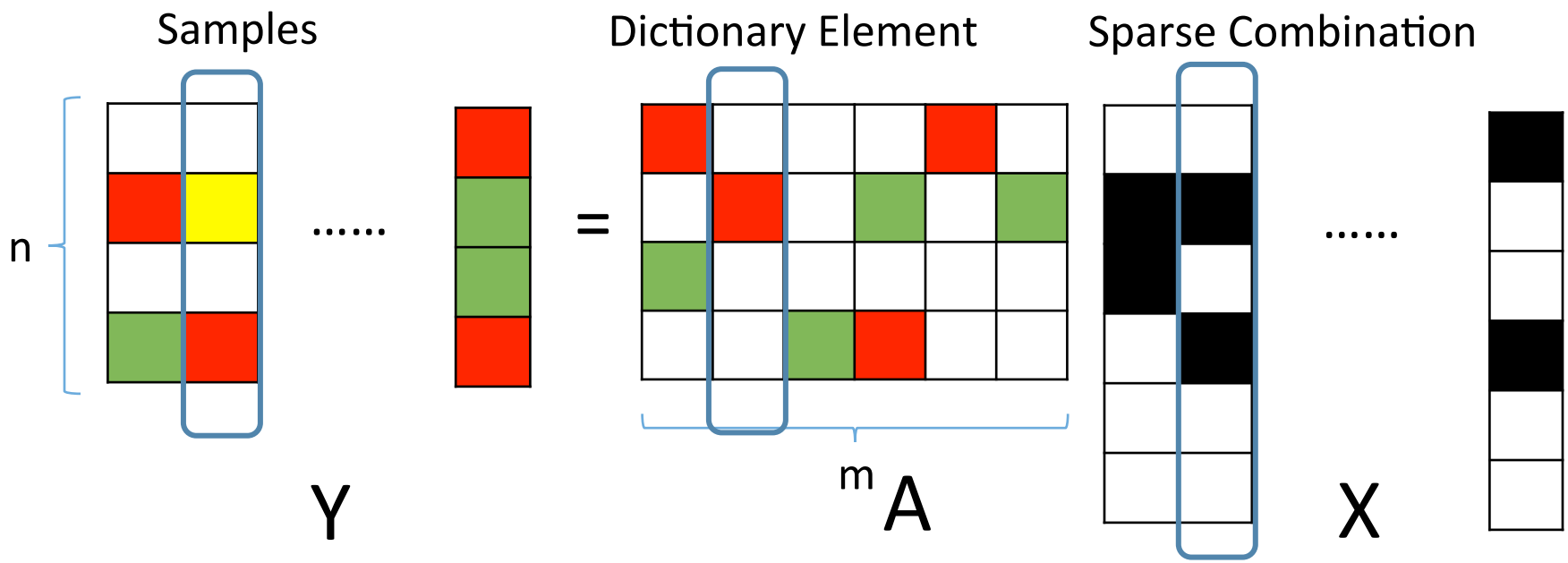


- Refining the solution

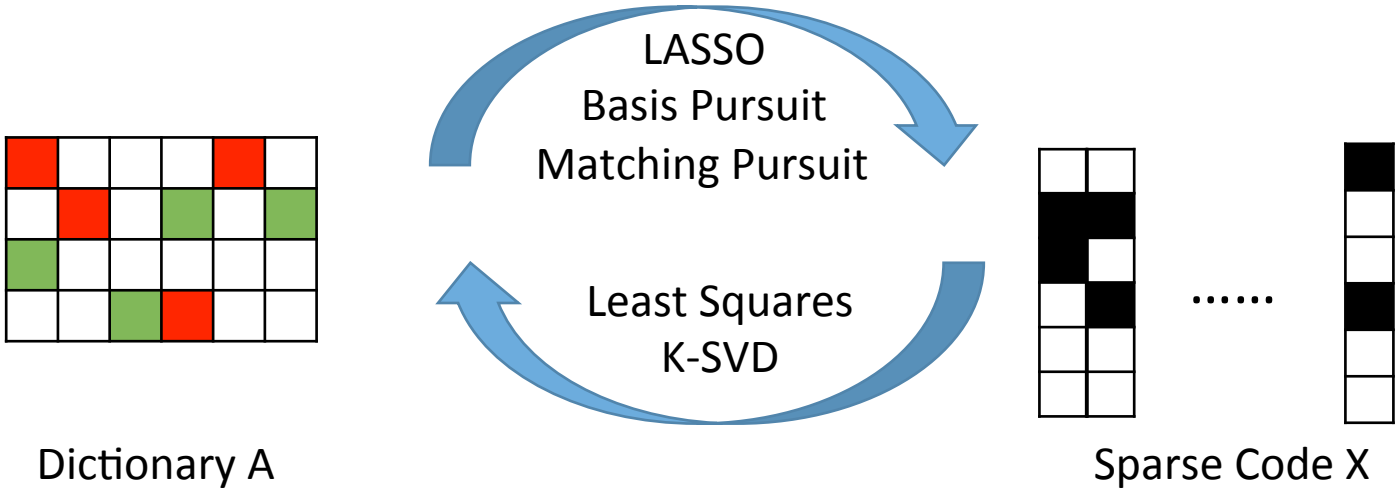


Dictionary Learning Problem

- Given samples of the form $Y = AX$
- X is a sparse matrix
- Goal: Learn A (dictionary).
- Interesting case: $m > n$ (overcomplete)

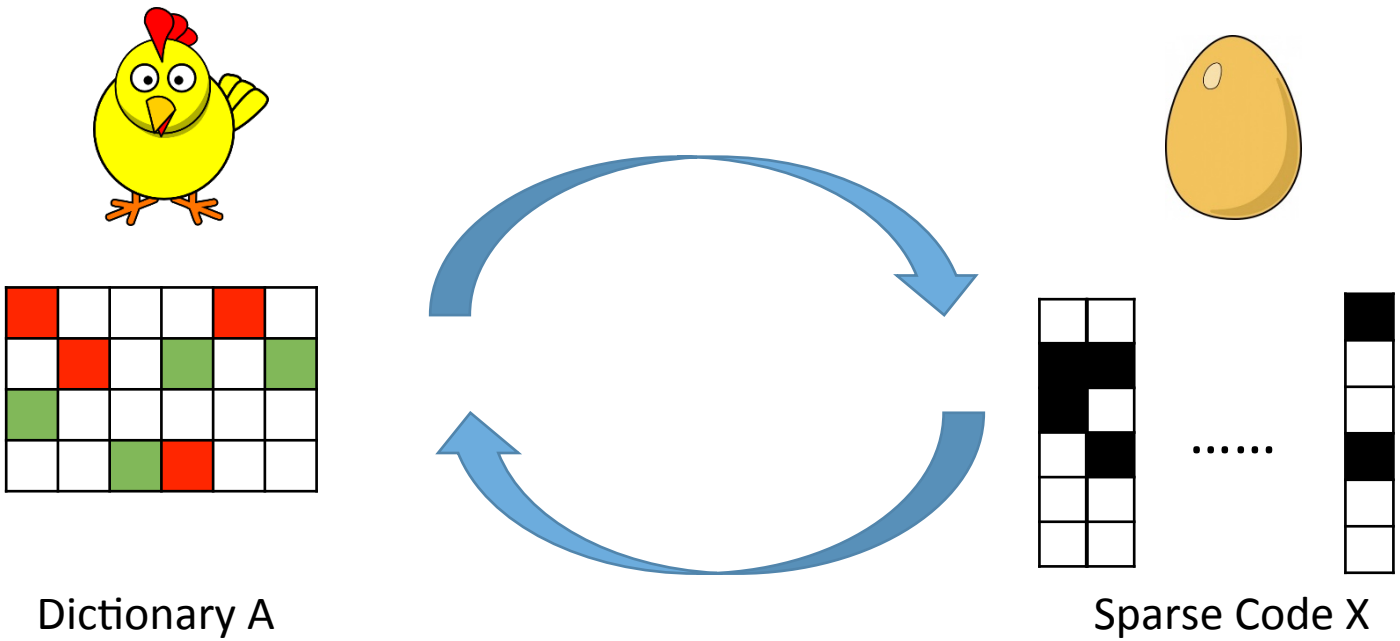


Previous Approach

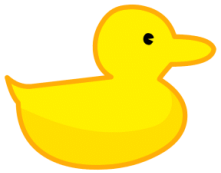


Alternating Minimization

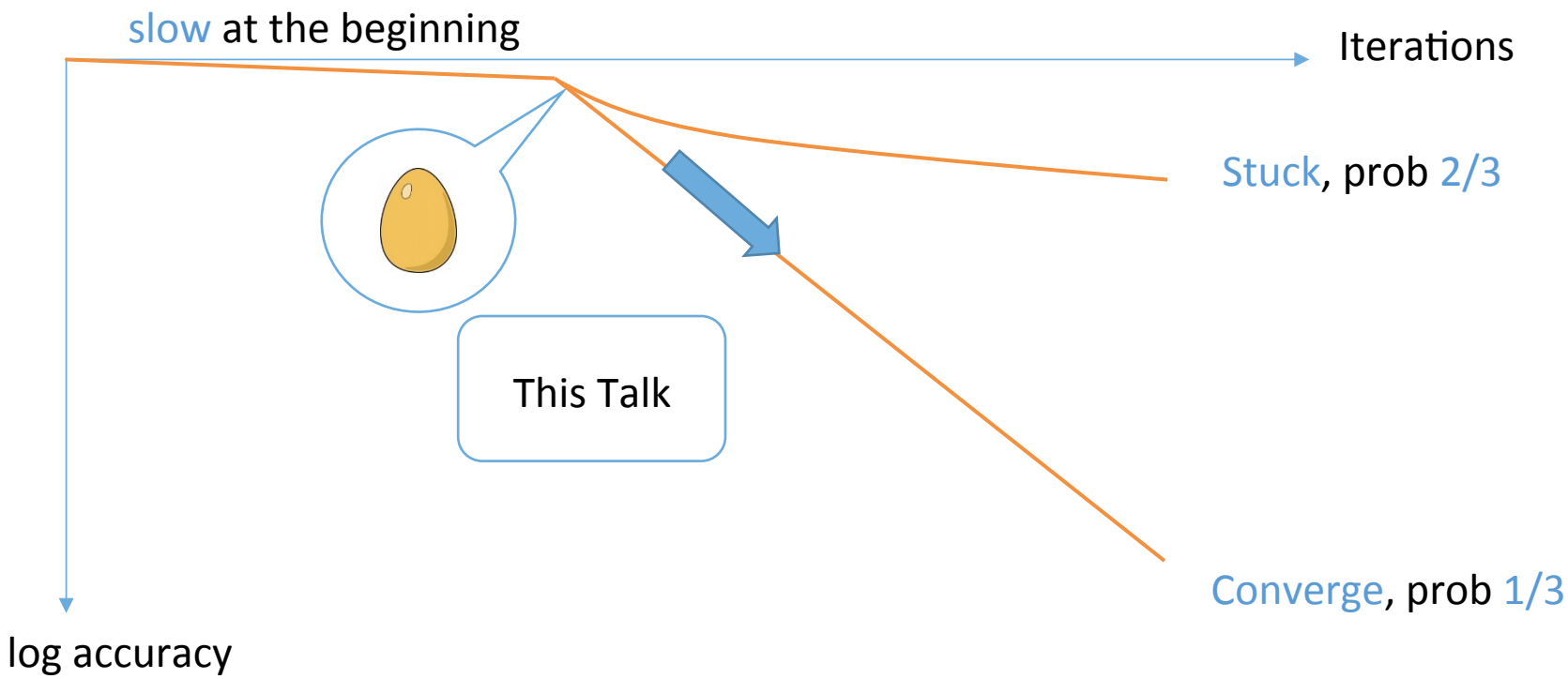
Problem with Alternating Minimization



- Monotone objective function?
  - Local Minimum Issues

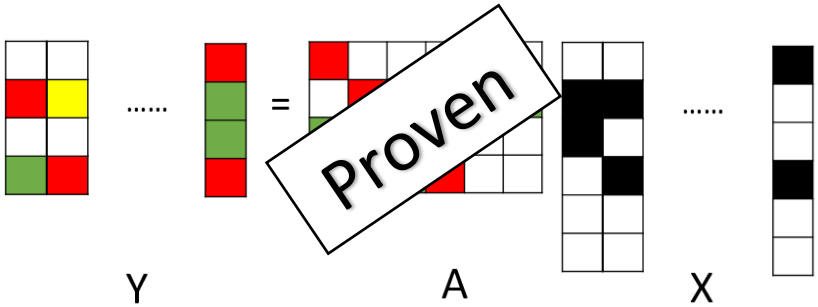


Empirical Behavior



- Synthetic Experiment, "qualitative" plot
- K-SVD converges with prob. 1/3, random samples as initial dict.

Provable Algorithms



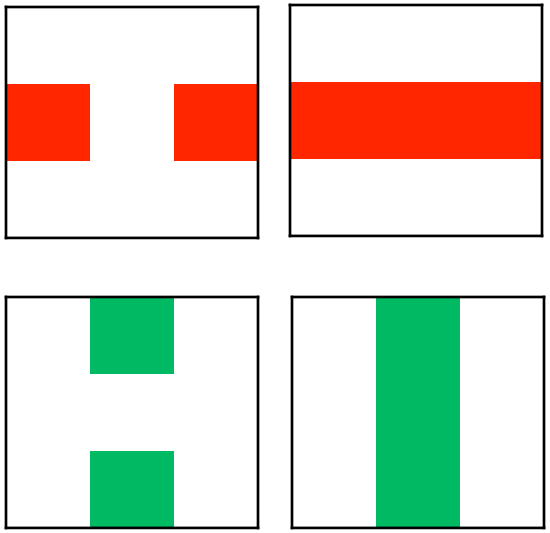
- Run in poly time, uses poly samples, learn the ground truth

- Separate modeling and optimization error
- Design new algorithms/Tweak old algorithms

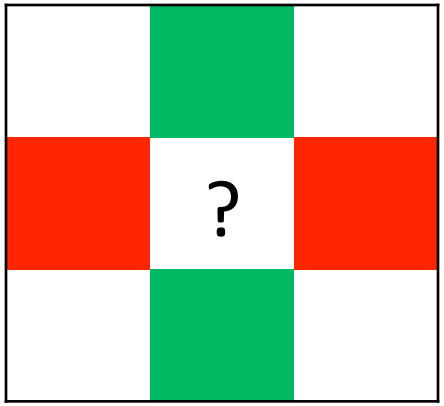
- Work only on "reasonable instances"

When is the solution “reasonable”?

- Consider Image Completion



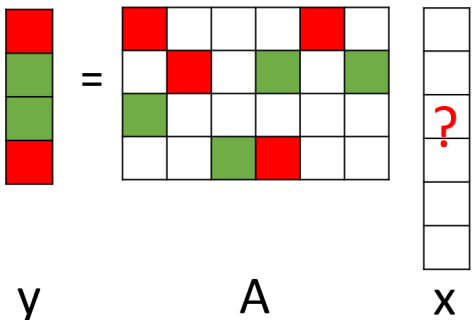
Dictionary



Image

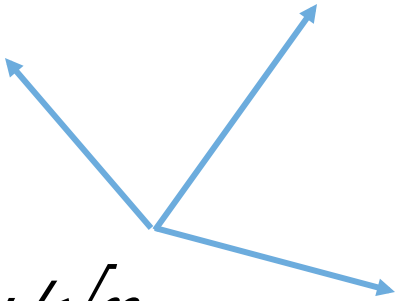
- The representation should be unique and robust!

Sparse Recovery



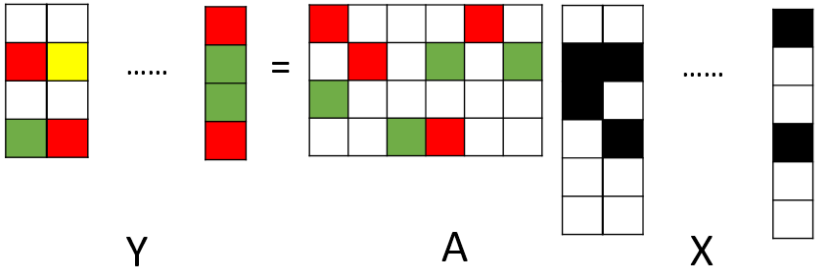
- Given A, y, find x.

- Incoherence [Donoho Huo '99]



- Dictionary elements have inner-product $\mu/\sqrt{n}$
- Solution is unique and robust
- Long line of work [Logan, Donoho Stark, Elad, …….]
- Sparsity up to $\sqrt{n}$

Our Results



- Thm: If dictionary A is incoherent,
  X is randomly k-sparse from "nice distribution",
  learn dictionary A with accuracy ε when sparsity

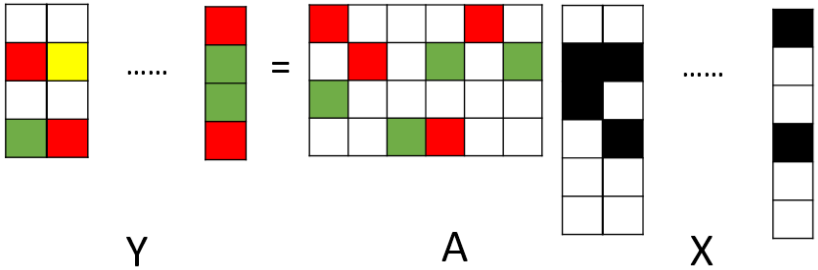
$$k \leq \min\{\sqrt{n}/\mu \log m \ , m^{0.4}\}$$

- Handles sparsity up to $\sqrt{n}$
- Sample complexity $O^*(m/\varepsilon^2)$.

Independently [Agarwal et al.] obtain similar result with slightly different assumptions and weaker sparsity

Later [Barak et al.] get stronger result using SOS

Our Results



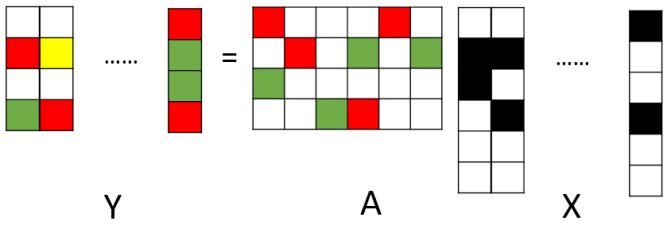
- Thm: Given an estimated dictionary ε-close to true dictionary, one iteration of K-SVD outputs a ε/2-close dictionary

- Works whenever ε<1/log m (before require 1/poly).
- Sample complexity O(mlog m)

- Combine: Can learn an incoherent dictionary with O*(m) samples in poly time.

Outline

- Dictionary Learning problem



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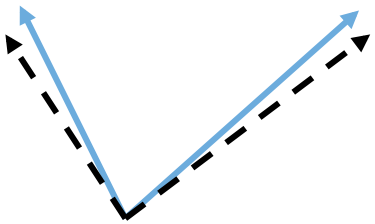

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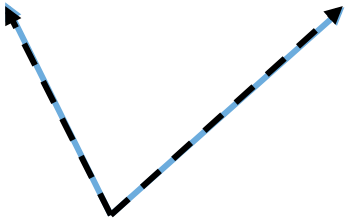
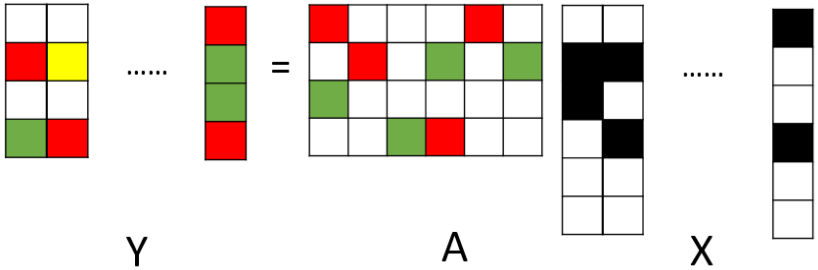


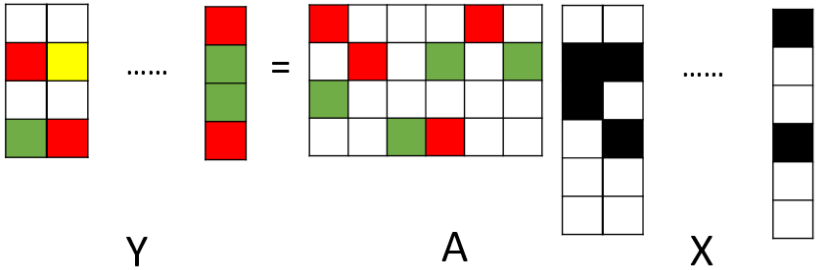
Ideas



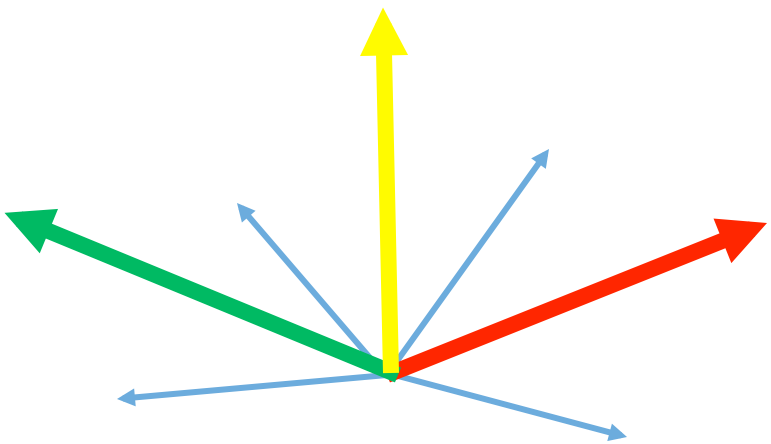
- Find the support of X, without knowing A.

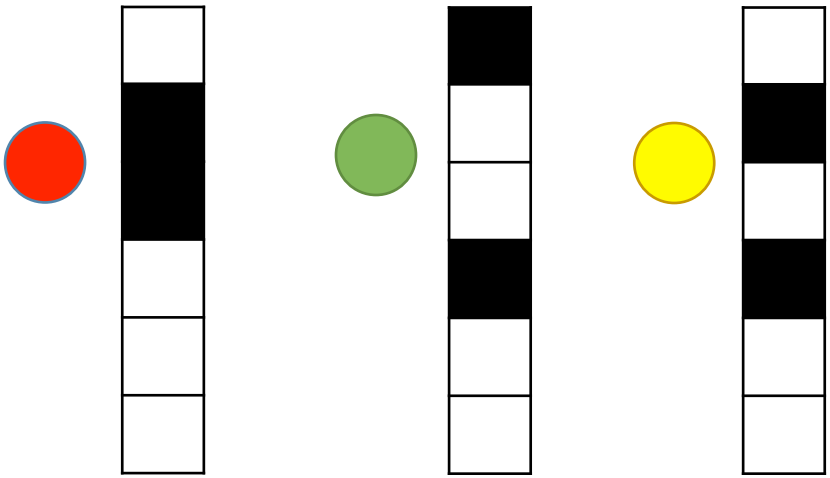
- Given support of X, find approximate A

Finding the Support



- Tool: Test whether two columns of X intersect



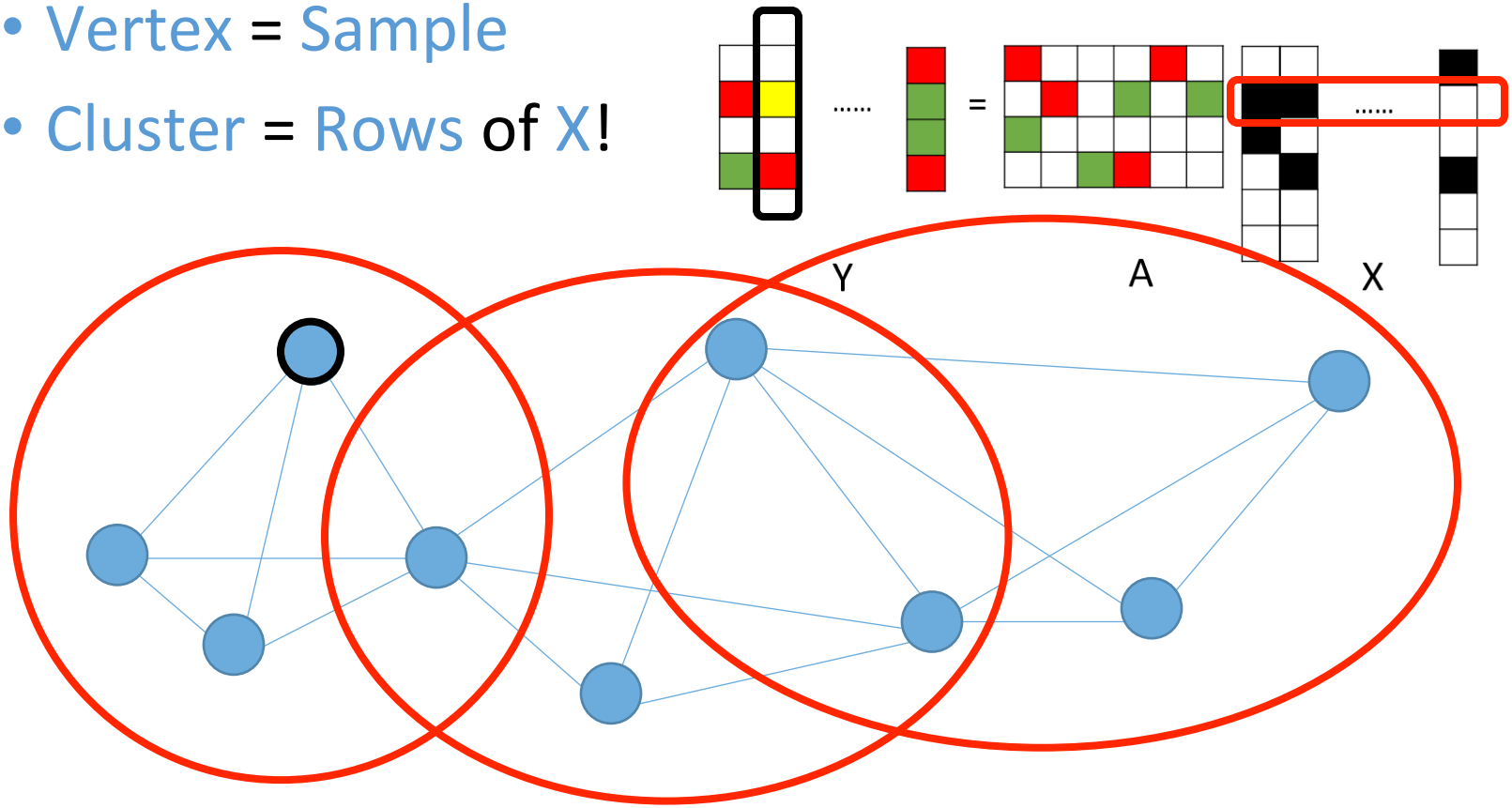


Disjoint ≈ Small Inner-product

Intersect ≈ Large Inner-product

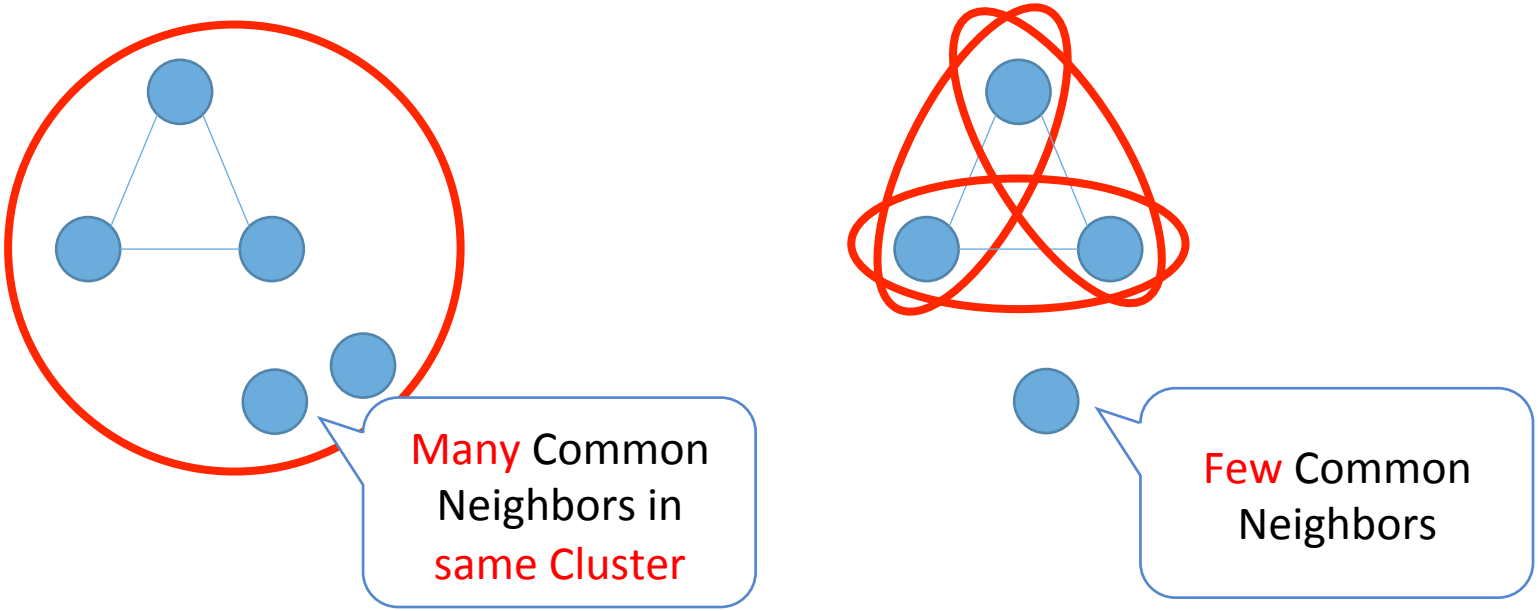
Finding the Support: Overlapping Clustering

- Connect pairs of samples with large inner-product
- Vertex = Sample
- Cluster = Rows of X!



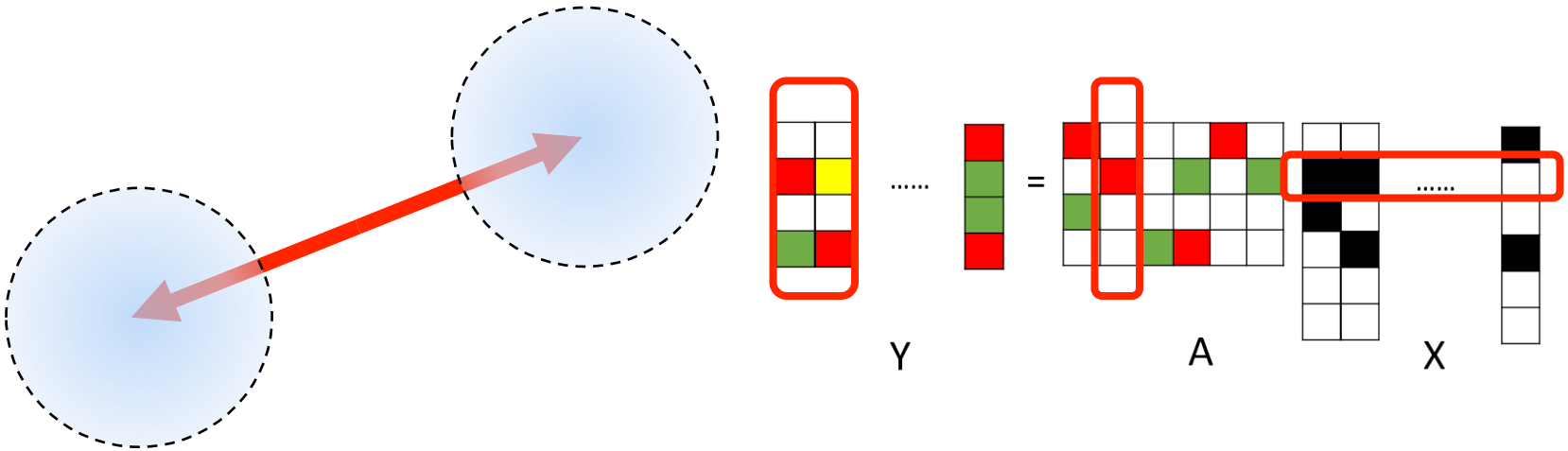
Overlapping Clustering

- Main problem



- Idea: Count the number of common neighbors
- pair of points share unique cluster ➔ cluster

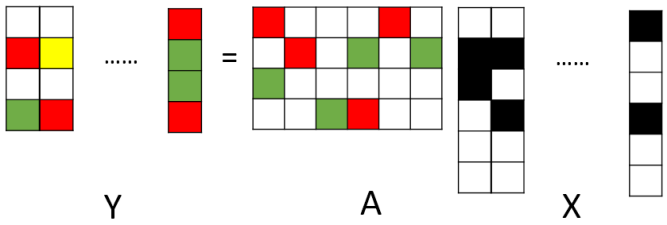
Estimate Dictionary Elements



- Focus on a row of X/column of A
  - Can use SVD to find maximum variance direction
  - Or take samples with same sign and average

Outline

- Dictionary Learning problem



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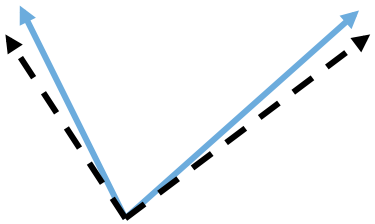
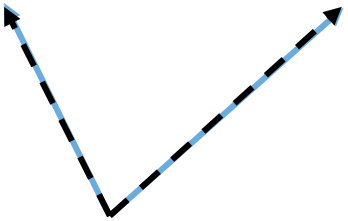


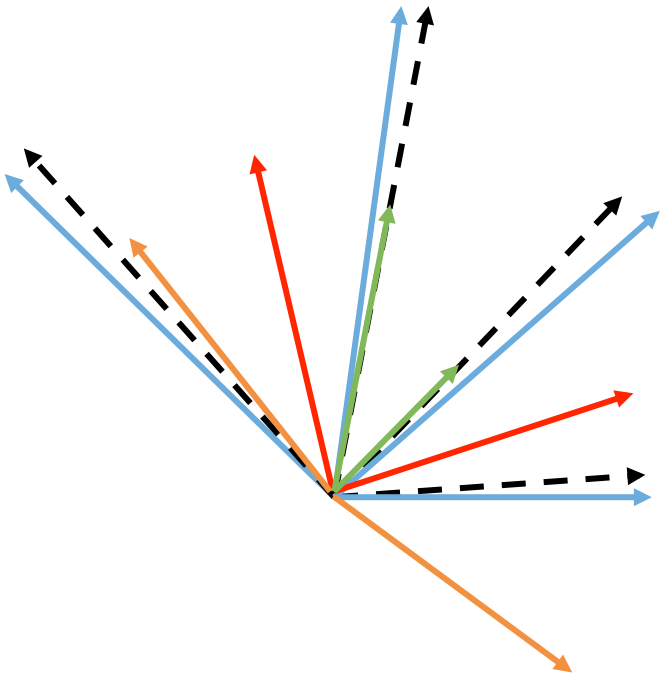
- Alternating Minimization Works!



K-SVD[Aharon,Elad, Bruckstein 06]

- Given: a good guess (A)
- Goal: find a even better dictionary
- Update one dict. element:
  - Take all samples with the element
  - Decode: $y \approx A\,x$
  - Residual: $r = y - \sum_{j \neq i} A_j\, x_j = \underbrace{\pm A_i + \sum_{j \neq i} (A_j\, x_j - A_j\, x_j)}_{\text{Noise}}$
  - Use top singular vec of residuals

K-SVD illustrated



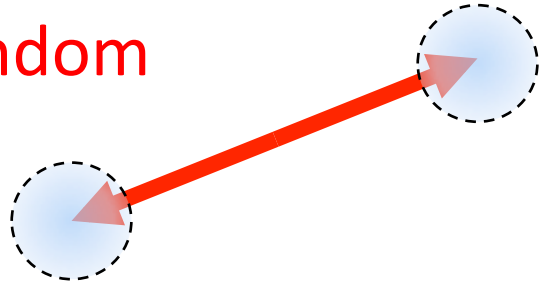
Blue: True dictionary
Dashed: Estimated Dictionary

Take all samples with same element
Compute Residual

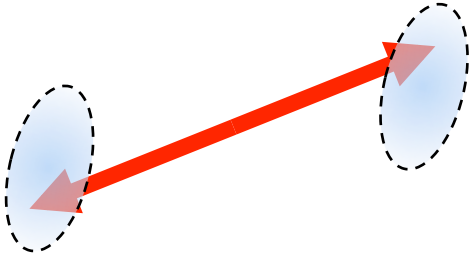
Hope: In residuals, noise is small and random,
top singular vector robust for random noise

K-SVD: Intuition

- When error ($A\downarrow i - A\downarrow i$) is random
  - Still incoherent
  - Can "decode"
  - Noise looks random



- When error is adversarial
  - May not be incoherent
  - Noise can be correlated



- Bad case: error is highly correlated, pointing to same direction

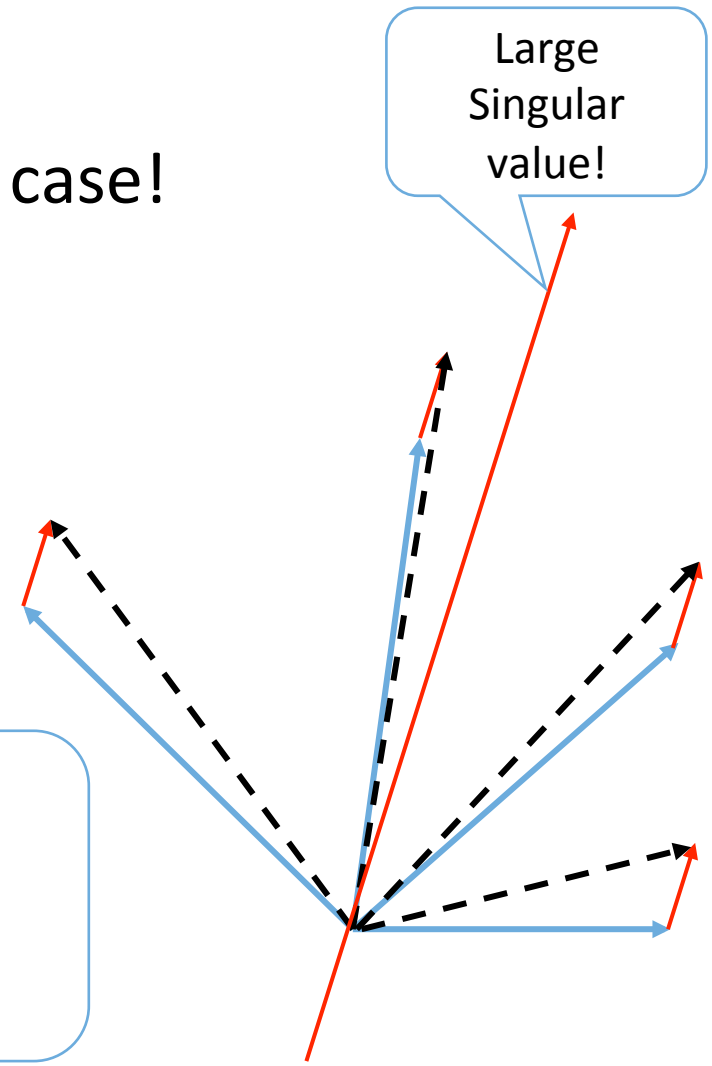
make the noise "random"

- Observation: Can detect the bad case!

- To handle bad case, need to
  - Perturb the estimated dictionary
  - Keep perturbation small
  - The result has low spectral norm

- $\min \|B\| \ @s.t. \ \|B\downarrow$

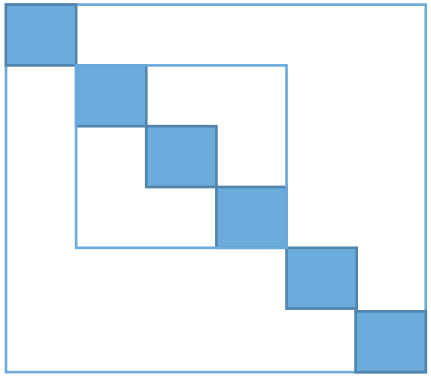
Convex!
OPT≤||A||



Low spectral norm is enough

- Key Lemma: When B has small spectral norm, $|\langle B_i, B_j \rangle| \leq 1/\log m$, random k columns of B are "almost orthogonal"
- ➔ Decoding is accurate for a random sample

- Proof sketch: For $B^T B$
  - Diagonals are large
  - Off-diagonals are small in Exp.
  - Concentration ➔ random submatrix is diag. dominant



Conclusion & Open Problems

- K-SVD works provably with good initialization
- Does the proof give any insight in practice?
  - Whitening

- "Error behaves random" useful in other settings?
- Handle larger sparsity?
  - work with RIP assumption?
- Lowerbounds?

Thank you!

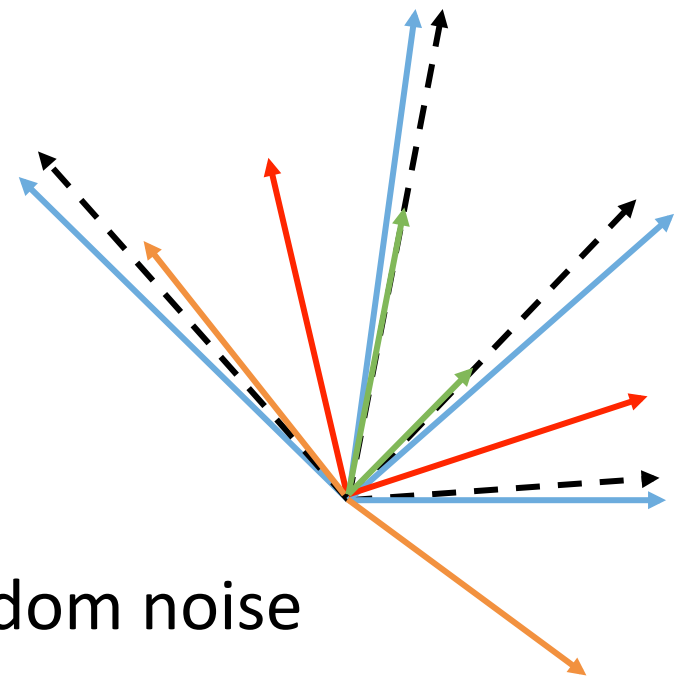
Thank you!

Questions?

K-SVD[Aharon,Elad, Bruckstein 06]

- Given: a good guess
- Goal: find a even better dictionary

- Update one dict. element:
  - Take all samples with the element
  - Remove other elements
  - Use top singular vec of residuals



- Hope: In residuals, error is small and random, SVD robust for random noise

Other Applications

Image Denoising [Mairal et al. ’09]

Digital Zooming [Couzinie-Devy ’10]

Applications

Image Completion [Mairal, Elad & Sapiro ’08]

Image Denoising [Mairal et al. ’09]

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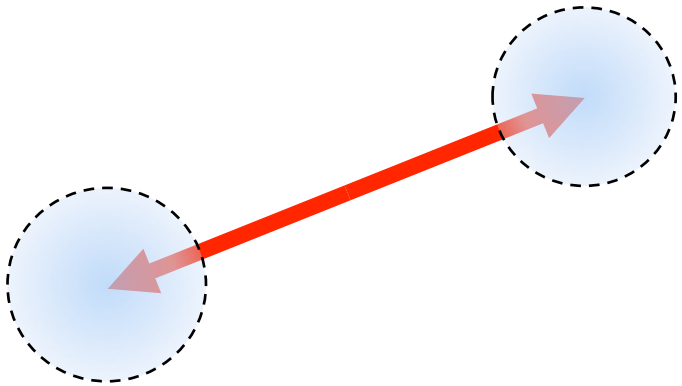
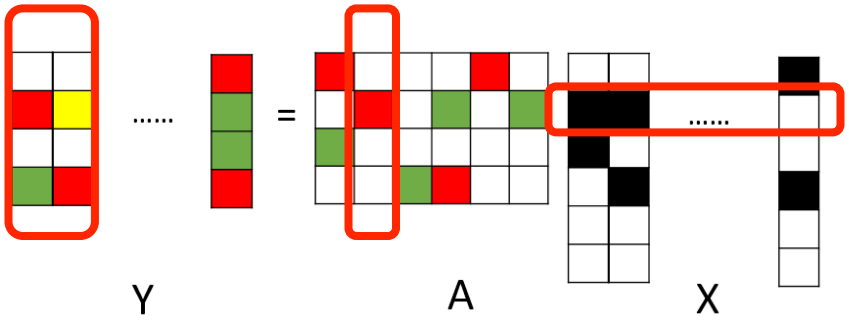
Refining the solution





- Use other columns to reduce the variance!
- Get ε accuracy with poly(m,n)log 1/ε samples